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RESONANT AND FLUORESCENT SCATTERING

IN PLANETARY ATMOSPHERES

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ABSTRACT

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Resonant and fluorescent scattering in plane-parallel planetary atmospheres is studied. Chandrasekhar's quadrature approximation method is extended to the radiative transfer equations of resonant and fluorescent emissions with coupling among transitions to obtain their intensities at arbitrary depth and direction in planetary atmospheres. The theory is applied for obtaining line intensities in the cases where the approximations of the single scattering and the assumption of semi-infinite atmosphere are valid respectively.

AUTHOR

INTRODUCTION

Resonant and fluorescent scattering of solar radiation is one of the important mechanisms of formation of emission lines and bands in planetary atmospheres. In the earth's upper atmosphere it is actually observed as day- or twilight- airglow. Planetary albedos may contain some contribution from resonant and fluorescent components. In many cases it is difficult to observe such resonant or fluorescent emissions, since they are usually hidden under the strong background continuum due to ground reflection and Rayleigh scattering in the lower dense atmosphere. This is the reason why the measurements of telluric sodium and H_α emissions are restricted during the twilight time. However, recent developments in space vehicles enable us to measure dayglow emissions in the high altitudes where they are free from contamination by scattered light. Other possibilities would be to observe resonant and fluorescent emissions in the middle and far-ultraviolet regions where various absorptive constituents of the planetary atmosphere prevent the reflections of continuum from planetary surface and underlying dense atmosphere. In such regions the resonance and fluorescence from the upper atmosphere can be detected even looking towards the planet during daytime.

Such a fortuitous condition may be found for the Schumann-Runge dayglow of molecular oxygen in the middle-ultraviolet. In the earth's atmosphere the height at which fluorescence may be significant falls between 30 km and 110 km. On the other hand, atmospheric ozone around the 30 km level strongly absorbs the ultraviolet radiations near 2500 Å, so that the ground reflection and Rayleigh scattering of solar ultraviolet radiation from underlying layers is effectively prevented from contaminating the Schumann-Runge dayglow. (Barth, 1963; Barth and Tohmatsu, 1963).

Accordingly, one may approach the mechanisms of planetary resonance and fluorescence from two different directions (Cf. Fig. 1):

(1) Resonant or fluorescent spectra as seen at the top and bottom of the scattering layer;

(2) Radiation intensity and emissivity at a particular point inside the layer.

The first category will be found in the work by Chamberlain and Sobouti (1962). They showed that the problem of planetary fluorescence can be formulated in the light of the theory of diffuse reflection from a plane-parallel atmosphere of finite or semi-infinite optical thickness. The solutions are then obtained in terms of H-, X- and Y- functions and their derivatives. This approach will have a wide applicability in studying planetary fluorescent albedos and telluric dayglow. On the other hand, the latter category which is more general in character, is of greater importance for interpreting the results of rocket photometry of dayglow emissions. This kind of approach will be suitable for deducing the altitude distribution of a fluorescent species from its intensity variation with height.

It may be readily observed that the basic ideas of the above two categories are closely related to each other, as one can see in Chandrasekhar's treatise on Radiative Transfer (1950) referred to as R. T. in this paper. According to Chandrasekhar, the source function and specific intensity can be approximated to arbitrary accuracy by using the quadrature expansion technique, and the H, X- and Y- functions are explicitly given as ultimate solutions in the limit of infinite approximation. This procedure may be applied also to more complicated planetary resonance and fluorescence problems. One complication, which is important in an actual problem, is the coupling of transitions as discussed by Sobouti (1962). The effect of coupling appears when the fluorescent system has more than one ground state. Since the temperature of planetary atmospheres is too low for thermal excitation of the electronic transitions, most of the couplings will be found in the vibrational and rotational structures of molecules and in spin substates of atoms.

Sobouti (1962) developed the theory of diffuse reflection and transmission with coupling among transitions and showed that the expressions for diffusely reflected and transmitted intensities can be formulated in terms of generalized H-, X- and Y- functions. His study essentially belongs to category (1), as stated above. The corresponding problem, which deals with the effect of coupling among transitions, still exists for category (2). The first section of this paper corresponds to this problem. There we shall discuss radiative transfer in a coupled resonant and fluorescent system for rather idealized cases. In Sec. II it will be shown that the case where there is continuous absorption can be converted to the standard problem by changing variables and constants. Sometimes we are more interested in the line or band intensity integrated over frequency rather than the line contour itself. It is often true that the observational data do not have sufficient accuracy to deduce line shapes. Sections III and IV are the theories of line intensity in the single scattering approximation and for the semi-infinite atmosphere respectively. This study corresponds to the theory of stellar absorption lines for the resonant and fluorescent emissions. Sec. V corresponds to the case where the excited state of a fluorescent system is populated by processes other than the resonant excitations from lower states; that is, the case of a planetary atmosphere with internal excitation sources. They may be either chemical or thermal excitations or cascading transitions from higher energy levels.

I. Basic Equations of Radiative Transfer for Coupled Resonant and Fluorescent Lines and Their Solutions

1. Basic Equations

Let us consider a group of the lines which arise in the absorption processes from the ground state $X_m \in X$, $(m = 1, 2, \ldots, s)$ to the excited state $A_p \in A$ which is common to all of the X_m's and the emission processes from A_p to the final state $B_\ell \in B$, $(\ell = 1, 2, \ldots, s)$. The line is called either resonant or fluorescent according to whether $B_\ell \in X$ or $B \notin X$ (cf. Fig. 2).

For mathematical convenience we shall make no distinction between the states belonging X and B, namely we put $B \equiv X$. This procedure does not specialize the problem, since the fluorescent radiations are free from the reabsorption process so that they may be treated as the "resonant lines with zero absorption coefficient". Then, since the excited state A_p is fixed, one can specify the quantities related to the transition $X_m \to A_p \to X_\ell$ by using only one suffix $\underline{m}$ or ℓ. For instance, the absorption coefficient and transition probability for the transition, $X_m \to A_p$ are written as κ_m and A_m respectively; likewise, the specific intensity for the transition, $A_p \to X_\ell$, as I_ℓ.

In formulating the radiative transfer equations, the following conditions will be assumed initially:

(i) The scattering is isotropic and coherent;

(ii) The scattering atmosphere is passive to the external radiation sources, and has no internal radiation sources beside the coherent re-emission of the absorbed radiation;

(iii) The excited state A_p can be populated only through absorption processes $X_m \to A_p$, $(m = 1, 2, \ldots, s)$, but not through the cascade transitions from the higher excited levels;

(iv) There is no continuous absorption other than the pertinent resonant absorption.

The cases where these conditions are not satisfied will be discussed in the later sections. In the plane-parallel atmosphere, the radiation field is governed by a set of integro-differential equations (Sobouti, 1962),

$$-\mu \frac{dI_\ell^\nu(t^\nu,\mu)}{dt} + k_\ell^\nu I_\ell^\nu(t^\nu,\mu) =$$
$$= \frac{\varpi_\ell^\nu}{2}\sum_{m=1}^{s} k_m^\nu \int_{-1}^{+1} I_m^\nu(t^\nu,\mu')\,d\mu' + \frac{\varpi_\ell^\nu}{4}\sum_{m=1}^{s} k_m^\nu F_m^\nu e^{-\frac{k_m^\nu}{\mu_0}t^\nu}, \tag{1}$$

where

$$t^\nu = \sum_{m=1}^{s} t_m^\nu = \sum_{m=1}^{s} \int_z^\infty \kappa_m^\nu n_m(z)\,dz, \tag{2}$$

is the "grand" optical depth, κ_m, $n_m(z)$ and z being the molecular absorption coefficient, number density in the m-th state, and height respectively;

$$k_m^\nu = \frac{\kappa_\ell^\nu}{\sum_{m=1} \kappa_m^\nu}, \tag{3}$$

the specific absorption coefficient of the ℓ-th component;

$$\varpi_\ell^\nu = \frac{A_\ell^\nu}{\sum_{m=1} (A_m^\nu + d_m)}, \tag{4}$$

the albedo of the ℓ-th component, A_ℓ, A_m and d_m being Einstein's transition probabilities and deactivation rate of the A_p state respectively:

$$\mu_0 = |\cos\theta_0|, \quad \mu = \cos\theta, \tag{5}$$

directional cosines of the incident flux and observing line of sight respectively;

$$\pi F_m^{\nu}, \tag{6}$$

incident radiation flux at the top of the atmosphere;

and

$$I_{\ell}^{\nu}(t^{\nu}, \mu), \tag{7}$$

the specific intensity of the ℓ-th component at depth, t^{ν} and in the direction, μ.

In this paper we will use the following purely non-dimensional equations

$$-\mu \frac{dI_{\ell}^{\nu}(t^{\nu},\mu)}{dt} + k_{\ell}^{\nu} I_{\ell}^{\nu}(t^{\nu},\mu) = = \frac{\varpi_{\ell}^{\nu}}{2} \sum_{m=1}^{s} k_m^{\nu} \int_{-1}^{+1} I_m^{\nu}(t^{\nu},\mu')\,d\mu' + \varpi_{\ell}^{\nu} \sum_{m=1}^{s} k_m^{\nu} f_m^{\nu}\, e^{-\frac{k_m^{\nu}}{\mu_0} t^{\nu}}, \quad (\ell = 1, 2, 3, \cdots, s), \tag{8}$$

where

$$f_m^{\nu} = \frac{\pi F_m^{\nu}}{\sum_{m=1}^{s} \pi F_m^{\nu}}, \tag{9}$$

is the specific incident flux for the m-th component, and $I_{\ell}^{\nu}(t^{\nu},\mu)$ is given in units of $\sum_{m=1}^{s} \pi F_m^{\nu}/4\pi$. Evidently, we have

$$\sum_{m=1}^{s} f_m^{\nu} \equiv 1, \tag{10}$$

One merit of using (8) for (1) will be that, if f_m^{ν} is replaced by πF_m^{ν}, in rayleighs/cm^2 / cm^{-1}, the solution will be obtained in units of rayleighs/cm^{-1}. Alternatively, if $f_n^{\nu} = 1$, while $f_{m \neq n}^{\nu} = 0$, one can obtain the response functions for the unit input of the n-th radiation because of the linearity of the equations. The suffix ν which indicates the wave number dependence of quantitites will hereafter be omitted so long as no confusion is possible.

In analog to Eq. (90) of R. T., Chapter I, the apparent solutions of (8) which satisfy the boundary conditions at both the top ($t = 0$) and bottom ($t = t_o$) of the scattering atmosphere are,

$$I_\ell(t, +\mu) = \int_t^{t_o} J_\ell(t') e^{-\frac{k_\ell}{\mu}(t'-t)} \frac{dt'}{\mu}, \tag{11a}$$

and

$$I_\ell(t, -\mu) = \int_0^{t} J_\ell(t') e^{-\frac{k_\ell}{\mu}(t-t')} \frac{dt'}{\mu},$$

$$(\ell = 1, 2, 3, \cdots, s; \quad 0 \leq \mu \leq 1), \tag{11b}$$

where $J_\ell(t)$ is the average intensity of the ℓ-th component (in units of $\sum_{m=1}^{s} \pi F_m / 4\pi$) at depth t, defined by

$$J_\ell(t) = \varpi_\ell \left(J_\ell^e(t) + J^s(t) \right), \tag{12}$$

with

$$J_\ell^e = \sum_{m=1}^{s} k_m f_m e^{-\frac{k_m}{\mu_o} t}, \tag{13}$$

and

$$J^{s}(t) = \frac{1}{2} \sum_{m=1}^{s} k_m \int_{-1}^{+1} I_m(t, \mu') d\mu'. \tag{14}$$

In the context of Chandrasekhar's quadrature approximation method, (R. T., p. 80), $J^s(t)$ may be approximated by a quadrature,

$$J^{s}(t) \simeq \frac{1}{2} \sum_{m=1}^{s} k_m \sum_{j=-n}^{+n} a_j I_{mj}, \tag{15}$$

where a_j and I_{mj} are respectively the Christoffel number and specific intensity which are associated to the characteristic root, μ_j, ($j = \pm 1, \pm 2, \ldots, \pm n$), of the Legendre polynominals, $P_{2n}(\mu_j) = 0$.

The following relations will be used frequently in the later discussions (R. T., p. 62):

$$a_j = a_{-j}, \quad (j = \pm 1, \pm 2, \cdots, \pm n); \tag{16}$$

$$\mu_{-j} = -\mu_j, \quad (j = \pm 1, \pm 2, \cdots, \pm n); \tag{17}$$

and

$$\sum_{j=-n}^{+n} a_j \mu_j^{g} = \frac{2\delta_{g,e}}{g+1}, \quad (g \leq 4n-1), \tag{18}$$

where

$$\delta_{g,e} = \begin{cases} 1, & \text{if } g \text{ is even} \\ 0, & \text{if } g \text{ is odd}. \end{cases} \tag{19}$$

Now, in terms of I_{mj}, Eq. (8) may be substituted by an equivalent set of 2sn linear differential equations,

$$-\mu_i \frac{dI_{\ell i}}{dt} + k_\ell I_{\ell i} = \frac{\varpi_\ell}{2} \sum_{m=1}^{s} k_m \sum_{j=-n}^{+n} a_j I_{mj} + \varpi_\ell \sum_{m=1}^{s} k_m f_m e^{-\frac{k_m}{\mu_0} t}, \tag{20}$$

Since these equations are linear with respect to $I_{\ell i}$, their solutions comprise one set of particular solutions and 2sn sets of the solutions for the associated homogeneous equations,

$$-\mu_i \frac{dI_{\ell i}}{dt} + k_\ell I_{\ell i} = \frac{\varpi_\ell}{2} \sum_{m=1}^{s} k_m \sum_{j=-n}^{+n} a_j I_{mj} \tag{21}$$

2. Particular Solutions

Since right hand side of Eq. (20) is regarded to be a product of and the "grand" source function,

$$\begin{aligned} J(t) &= J^{s}(t) + J^{e}(t) \\ &= \frac{1}{2} \sum_{m=1}^{s} k_m \sum_{j=-n}^{+n} a_j I_{mj} + \sum_{m=1}^{s} k_m f_m e^{-\frac{k_m}{\mu_0} t}, \end{aligned} \tag{22}$$

which is a function of only t, a particular solution of $I_{\ell i}$ is easily obtained as

$$I_{\ell i} = \frac{\varpi_\ell}{k_\ell} \sum_{m=1}^{s} k_m f_m \gamma_m \frac{1}{1 + \frac{\mu_i k_m}{\mu_0 k_\ell}} e^{-\frac{k_m}{\mu_0} t}, \tag{23}$$

where γ_m, (m = 1, 2, ..., s) is a constant defined by

$$\gamma_m = \frac{1}{1 - \sum_{n=1}^{s} \frac{\varpi_n}{2} \sum_{j=-n}^{+n} \frac{a_j}{1 + \frac{\mu_j k_m}{\mu_0 k_n}}} = \frac{1}{1 - \sum_{n=1}^{s} \varpi_n \sum_{j=+1}^{+n} \frac{1}{1 - \left(\frac{\mu_j k_m}{\mu_0 k_n}\right)^2}}. \tag{24}$$

This expression may be compared to R. T., p. 82, Eq. 90.

3. Solutions of the Associated Homogeneous System

The solutions of the associated homogeneous system, (21) are linear combinations of 2sn distinct fundamental solutions,

$$I_{li} = \frac{\varpi_l}{k_l} \frac{1}{1 + \frac{\mu_i k_\alpha}{k_l}} e^{-k_\alpha t},$$

$$(l = 1, 2, \cdots, s; \; i = \pm 1, \pm 2, \cdots, \pm n; \; \alpha = \pm 1, \pm 2, \cdots, \pm sn), \tag{25}$$

where

$$k_\alpha = -k_{-\alpha}, \tag{26}$$

stands for one of the roots of the associated Wronskian equation,

$$1 = \sum_{m=1}^{s} \frac{\varpi_m}{2} \sum_{j=-n}^{+n} \frac{a_j}{1 + \frac{\mu_j k_\alpha}{k_m}}, \tag{27}$$

or

$$1 = \sum_{m=1}^{s} \varpi_m \sum_{j=+1}^{+n} \frac{a_j}{1 - \left(\frac{\mu_j k_\alpha}{k_m}\right)^2}, \tag{27a}$$

(Cf. R. T., p. 81).

In fact, it may be shown that Eq. (27) allows 2sn distinct real solutions

$$k_\alpha = -k_{-\alpha}, \quad (\alpha = \pm 1, \pm 2, \cdots, \pm sn), \tag{28}$$

so that the general solutions of (21) can be given by 2sn linear combinations of (25); namely

$$I_{\ell i} = \frac{\varpi_\ell}{k_\ell} \sum_{\alpha=-sn}^{+sn} L_\alpha \frac{1}{1 + \frac{\mu_i k_\alpha}{k_\ell}} e^{-k_\alpha t}, \tag{29}$$

where L_α's are 2sn constants.

However, in the conservative case*, i.e., $\sum_{m=1}^{s} \varpi_m = 1$ while $k_m \neq 0$ for all m,

$$k_{\pm sn} = 0, \tag{30}$$

provide doubly degenerate zeros of (27).

In this case, it may be seen readily that

$$I_{\ell i} = \frac{\varpi_\ell}{k_\ell} b\left(t + \frac{\mu_i}{k_\ell}\right), \tag{31}$$

and

$$I_{\ell i} = \frac{\varpi_\ell}{k_\ell} Q, \tag{32}$$

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* We shall define as the conservative system that system in which all the components are resonant ($k \neq 0$) and $\sum_{m=1}^{s} \varpi_m = 1$ is satisfied. The system will not be conservative even if $\sum_{m=1}^{s} \varpi_m = 1$ when there exists any fluorescent component, since for this component, $k_m = 0$.

where b and Q are constants*, are two independent solutions to be added to other 2sn-2 solutions given as (29). Therefore, in the conservative case, the general solutions (29) should be replaced by

$$I_{\ell i} = \frac{\varpi_\ell}{k_\ell}\left[b\left(t + \frac{\mu_i}{k_\ell}\right) + Q + \sum_{\alpha=-sn+1}^{sn-1} L_\alpha \frac{1}{1 + \frac{\mu_i k_\alpha}{k_\ell}} e^{-k_\alpha t} \right]. \tag{31}^*$$

4. <u>Complete Solutions</u>

From (23) and (31) one can obtain the complete solutions of (20) for the conservative case as,

$$I_{\ell i} = \frac{\varpi_\ell}{k_\ell}\left[b\left(t + \frac{\mu_i}{k_\ell}\right) + Q + \sum_{\alpha=-sn+1}^{sn-1} L_\alpha \frac{1}{1 + \frac{\mu_i k_\alpha}{k_\ell}} e^{-k_\alpha t} + \sum_{m=1}^{s} k_m f_m \gamma_m \frac{1}{1 + \frac{\mu_i k_m}{\mu_0 k_\ell}} e^{-\frac{k_m}{\mu_0} t} \right]. \tag{32}$$

The solutions for non-conservative cases have the same form as (32) with b = Q = 0, and the summation over α extended from -sn to +sn. We shall hereafter refer to Eq. (32) as the standard solutions, since the non-conservative cases are dealt with in quite analogous fashion, merely dropping the terms with b and Q. This equation may be compared to R. T., p. 82, Eq. 92.

Substitution of (32) in (22) results in the grand source function.

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* Chandrasekhar excludes b outside of the brackets, [ ], so that his bQ and bL_α's are the present Q and L_α's respectively.

$$J(t) = bt + Q + \sum_{\alpha=-sn+1}^{sn-1} L_\alpha e^{-k_\alpha t} + \sum_{m=1}^{s} k_m f_m \gamma_m e^{-\frac{k_m}{\mu_0}t} . \tag{33}$$

(Cf. R. T., p. 196, Eq. 77)

5. Net Flux and Total Reflection and Transmission

Similarly, the net flux of the whole fluorescent system* is obtained (in units of $\sum_{m=1}^{s} \pi F_m$) as

$$\begin{aligned} \pi F_0(t) &= \sum_{\ell=1}^{s} 2\pi \int_{-1}^{+1} \frac{I_\ell(t,\mu')}{4\pi} \mu' d\mu' \\ &\simeq \left(\sum_{m=1}^{s} f_m\right) \cdot \frac{1}{2} \sum_{\ell=1}^{s} \sum_{i=-n}^{+n} I_{\ell i} \mu_i a_i \\ &= \frac{1}{3} b \sum_{\ell=1}^{s} \frac{\varpi_\ell}{k_\ell^2} + \left(\sum_{\ell=1}^{s} \varpi_\ell - 1\right) \sum_{\alpha=-sn+1}^{sn-1} \frac{L_\alpha}{k_\alpha} e^{-k_\alpha t} + \\ &\quad + \mu_0 \sum_{m=1}^{s} \left[1 + \left(\sum_{\ell=1}^{s} \varpi_\ell - 1\right) \gamma_m \right] f_m e^{-\frac{k_m}{\mu_0}t} . \end{aligned} \tag{34}$$

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* The external part of flux, $-\sum_{m=1}^{s} f_m e^{-\frac{k_m}{\mu_0}t}$ is excluded.

The total reflection from the top is then,

$$\begin{aligned}\pi F_{ref} &= +\pi F_0(0) \\ &= \frac{1}{3} b \sum_{\ell=1}^{s} \frac{\varpi_\ell}{k_\ell^2} + \left(\sum_{\ell=1}^{s} \varpi_\ell - 1\right) \sum_{\alpha=-sn+1}^{sn-1} \frac{L_\alpha}{k_\alpha} + \\ &\quad + \mu_0 \sum_{m=1}^{s} \left[1 - \left(\sum_{\ell=1}^{s} \varpi_\ell - 1\right) \gamma_m \right] f_m . \end{aligned} \tag{35}$$

Likewise, the total transmission from the bottom is

$$\begin{aligned}\pi F_{trans} &= -\pi F_0(t_0) \\ &= -\frac{1}{3} b \sum_{\ell=1}^{s} \frac{\varpi_\ell}{k_\ell^2} - \left(\sum_{\ell=1}^{s} \varpi_\ell - 1\right) \sum_{\alpha=-sn+1}^{sn-1} \frac{L_\alpha}{k_\alpha} e^{-k_\alpha t_0} \\ &\quad - \mu_0 \sum_{m=1}^{s} \left[1 - \left(\sum_{\ell=1}^{s} \varpi_\ell - 1\right) \gamma_m \right] f_m e^{-\frac{k_m}{\mu_0} t_0} . \end{aligned} \tag{36}$$

Especially for the conservative system, since $\sum_{\ell=1}^{s} \varpi_\ell \equiv 1$, one obtains

$$\pi F_{ref} = \frac{1}{3} b \sum_{\ell=1}^{s} \frac{\varpi_\ell}{k_\ell^2} + \mu_0 \sum_{m=1}^{s} f_m \tag{37}$$

and

$$\pi F_{trans} = -\frac{1}{3} b \sum_{\ell=1}^{s} \frac{\varpi_\ell}{k_\ell^2} - \mu_0 \sum_{m=1}^{s} f_m e^{-\frac{k_m}{\mu_0} t_0} . \tag{38}$$

Whence

$$\pi F_{ref} + \pi F_{trans} = \mu_0 \sum_{m=1}^{s} f_m \left(1 - e^{-\frac{k_m}{\mu_0} t_0}\right) \tag{39}$$

This relation implies that the total amount of radiation emergent from the layer balances exactly the total amount of radiations absorbed.

Further, since both πF_{ref} and πF_{trans} are positive and never exceed the total amount of radiation absorbed we have from (37), (38) and (39).

$$3\mu_0 \frac{\sum_{m=1}^{s} f_m e^{-\frac{k_m}{\mu_0} t_0}}{\sum_{\ell=1}^{s} \frac{\varpi_\ell}{k_\ell^2}} \leq |b| \leq 3\mu_0 \frac{\sum_{m=1}^{s} f_m}{\sum_{\ell=1}^{s} \frac{\varpi_\ell}{k_\ell^2}} \tag{40}$$

and

$$b \leq 0 \tag{41}$$

6. Boundary Conditions

The 2sn constants, b, Q, and L_α's in Eq. (32) are to be determined so that $I_{\ell i}$'s satisfy the boundary conditions at two boundaries, namely,

$$[I_{\ell,-i}]_{t=0} = 0 ;$$

$$-b \frac{\mu_i}{k_\ell} + Q + \sum_{\alpha=-sn+1}^{sn-1} L_\alpha \frac{1}{1 - \frac{\mu_i k_\alpha}{k_\ell}} +$$

$$+ \sum k_m f_m \gamma_m \frac{1}{1 - \frac{\mu_i k_m}{\mu_0 k_\ell}} = 0 , \tag{42a}$$

and

$$[I_{\ell,i}]_{t=t_0} = 0 ;$$

$$b\left(t_0 + \frac{\mu_i}{k_\ell}\right) + Q + \sum_{\alpha=-sn+1}^{sn-1} L_\alpha \frac{1}{1 + \frac{\mu_i k_\alpha}{k_\ell}} e^{-k_\alpha t_0} +$$

$$+ \sum_{m=1}^{s} k_m f_m \gamma_m \frac{1}{1 + \frac{\mu_i k_m}{\mu_0 k_\ell}} e^{-\frac{k_m}{\mu_0} t_0} = 0 , \tag{42b}$$

$$(\ell = 1, 2, \cdots, s ; \; i = +1, +2, \cdots, +n).$$

7. Standard Solutions for Specific Intensities

Once the b, Q and L_α 's are determined by (42a, b), the specific intensities can be derived from the grand source function (33). Since

$$J_\ell(t) = \varpi_\ell J(t) = \varpi_\ell \left[bt + Q + \sum_{\alpha=-sn+1}^{sn-1} L_\alpha e^{-k_\alpha t} + \sum_{m=1}^{s} k_m f_m \gamma_m e^{-\frac{k_m}{\mu_0} t} \right], \tag{43}$$

integration of (11a, b) yields,

$$\begin{aligned} I_\ell(t,+\mu) = \frac{\varpi_\ell}{k_\ell} \Bigg\{ & b \left[\left(\frac{\mu}{k_\ell} + t \right) - \left(\frac{\mu}{k_\ell} + t_0 \right) e^{-\frac{k_\ell}{\mu}(t_0 - t)} \right] + \\ & + Q \left[1 - e^{-\frac{k_\ell}{\mu}(t_0 - t)} \right] + \\ & + \sum_{\alpha=-sn+1}^{sn-1} L_\alpha \frac{1}{1 + \frac{\mu k_\alpha}{k_\ell}} \left[e^{-k_\alpha t} - e^{-\left(\frac{k_\ell}{\mu} + k_\alpha\right) t_0 + \frac{k_\ell}{\mu} t} \right] + \\ & + \sum_{m=1}^{s} k_m f_m \gamma_m \frac{1}{1 + \frac{\mu k_m}{\mu_0 k_\ell}} \left[e^{-\frac{k_m}{\mu_0} t} - e^{-\left(\frac{k_\ell}{\mu} + \frac{k_m}{\mu_0}\right) t_0 + \frac{k_\ell}{\mu} t} \right] \Bigg\}, \end{aligned} \tag{44a}$$

and

$$\begin{aligned} I_\ell(t,-\mu) = \frac{\varpi_\ell}{k_\ell} \Bigg\{ & b \left[t - \frac{\mu}{k_\ell} \left(1 - e^{-\frac{k_\ell}{\mu} t} \right) \right] + \\ & + Q \left[1 - e^{-\frac{k_\ell}{\mu} t} \right] + \\ & + \sum_{\alpha=-sn+1}^{sn-1} L_\alpha \frac{1}{1 - \frac{\mu k_\alpha}{k_\ell}} \left[e^{-k_\alpha t} - e^{-\frac{k_\ell}{\mu} t} \right] + \\ & + \sum k_m f_m \gamma_m \frac{1}{1 - \frac{\mu k_m}{\mu_0 k_\ell}} \left[e^{-\frac{k_m}{\mu_0} t} - e^{-\frac{k_\ell}{\mu} t} \right] \Bigg\}. \end{aligned} \tag{44b}$$

(Cf. R. T., p. 83, Eq. 99)

II. Solution of Radiative Transfer for Coupled Resonant or Fluorescent Lines in the Presence of Continuous Absorption

If continuous absorption coexists with the selective resonant absorptions, equation (8) will be modified (Sobouti, 1962) to

$$-\mu \frac{dI_\ell(t',\mu)}{dt'} + K_\ell\, I_\ell(t',\mu) =$$
$$= \frac{\varpi_\ell}{2}\sum_{m=1}^{s} k_m' \int_{-1}^{+1} I_m(t',\mu')\,d\mu' + \varpi_\ell \sum_{m=1}^{s} k_m' f_m\, e^{-\frac{K_m}{\mu_0}t'} \tag{45}$$

where t', k_m' and K_ℓ are newly defined using the line absorption coefficients, $\varkappa_\ell$, (ℓ = 1, 2, ..., s) and continuous absorption coefficients, σ_ℓ, (= 1, 2, ..., s) by

$$t' = \sum_{m=1}^{s} \int_z^{\infty} (\varkappa_m + \sigma_m)\, n_m(z)\,dz \simeq \frac{\sum_{m=1}^{s} (\varkappa_m + \sigma_m)}{\sum_{m=1}^{s} \varkappa_m}\, t, \tag{46}$$

$$k_m' = \frac{\varkappa_m}{\sum_{m=1}^{s} (\varkappa_m + \sigma_m)}\,. \tag{47}$$

and

$$K_\ell = \frac{\varkappa_\ell + \sigma_\ell}{\sum_{\ell=1}^{s} (\varkappa_\ell + \sigma_\ell)}\,. \tag{48}$$

It can be shown that this planetary Milne-Eddington equation can be converted to the standard form, Eq. (8) by transformation of variables and constants. First of all, we put

$$\varpi_\ell{}' = \frac{k_\ell'}{K_\ell}\varpi_\ell . \tag{49}$$

$$f_\ell' = \frac{k_\ell'}{K_\ell} f_\ell , \tag{50}$$

$$I_\ell'(t',\mu') = \frac{k_\ell'}{K_\ell} I_\ell(t',\mu') , \tag{51}$$

and

$$J_\ell'(t',\mu') = \frac{k_\ell'}{K_\ell} J_\ell(t',\mu) \tag{52}$$

Equation (49) may be compared with Eq. (2) of Chamberlain and Sobouti (1962).

Now, it will be observed that Eq. (45) is rewritten in terms of the above quantities as

$$-\mu\frac{dI_\ell'(t',\mu)}{dt'} + K_\ell I_\ell'(t',\mu) = \\ = \frac{\varpi_\ell'}{2}\sum_{m=1}^{s} K_m \int_{-1}^{+1} I_m'(t',\mu')\,d\mu' + \varpi_\ell'\sum_{m=1}^{s} K_m f_m' e^{-\frac{K_m}{\mu_0}t'} . \tag{53}$$

This equation has exactly the same form as Eq. (8) which is derived for an atmosphere without continuous absorption, if the variables and constants with a prime in Eq. (53) correspond to those without a prime in Eq. (8), and K_ℓ to k_ℓ. This result means that all the "monochromatic" relations developed in the previous section, for example, equations (44a, b) remain valid with the above new variables and constants even when allowing for the existence of continuous absorption.

Therefore, solutions are first obtained in this new variable system and then they are converted to the quantities of the real radiation field by referring to (49) - (52).

III. Line Intensities in the Single Scattering Approximation

The results of Sec. I can be reduced to simpler forms in cases where either the albedos of the "resonant" components are small, or the total optical thickness is small. In both cases the solutions have the same form as (11a, b) where $J_\ell(t)$ is approximated by

$$J_\ell(t) \simeq \varpi_\ell J^e(t) = \varpi_\ell \sum_{m=1}^{s} k_m f_m \, e^{-\frac{k_m}{\mu_0} t}, \tag{54}$$

This approximation corresponds to having

$$\gamma_m \to 0, \quad \text{and} \quad b = Q = L_\alpha \to 0,$$

in (44a, b). Accordingly we have

$$I_\ell(t,+\mu) = \frac{\varpi_\ell}{k_\ell} \sum_{m=1}^{s} k_m f_m \frac{1}{1+\frac{\mu k_m}{\mu_0 k_\ell}} \left[e^{-\frac{k_m}{\mu_0} t} - e^{-\left(\frac{k_m}{\mu_0}+\frac{k_\ell}{\mu}\right)t_0 + \frac{k_\ell}{\mu} t} \right], \tag{55a}$$

$$I_\ell(t,-\mu) = \frac{\varpi_\ell}{k_\ell} \sum_{m=1}^{s} k_m f_m \frac{1}{1-\frac{\mu k_m}{\mu_0 k_\ell}} \left[e^{-\frac{k_m}{\mu_0} t} - e^{-\frac{k_\ell}{\mu} t} \right]. \tag{55b}$$

(Cf. R. T., p. 217, Eq. 60)

Although the solutions for both $\sum_{Res.} \varpi_\ell \ll 1$ and $t_o \ll 1$, have the same form, the reason for the approximation procedure is different in the two cases. In the former case, the above solutions can be considered as the first order approximation of $I_\ell(t, \mu)$ when it is developed in power series of ϖ_ℓ as

$$I_\ell(t,\mu) = I^{(\ell)}(t,\mu)\varpi_\ell + \sum_{m=1}^{s} I^{(\ell m)}(t,\mu)\varpi_\ell\varpi_m + \cdots \tag{56}$$

On the other hand, the latter case is based on the assumption that

$$\frac{1}{2}\sum_{m=1}^{s} k_m \int_{-1}^{+1} I_m(t,\mu')d\mu' \ll \sum_{m=1}^{s} k_m f_m e^{-\frac{k_m}{\mu_0}t}, \tag{57}$$

for small values of t_o.

Quite often the total intensity of line component is the measurable quantity instead of the "monochromatic" intensity. Here the line intensity is defined by

$$\mathcal{J}_\ell(t,\mu) = \int_{Line} I_\ell^\nu(t,\mu)d\nu . \tag{58}$$

Integration of (55a, b) over ν can be done easily, when all the line components have the same Doppler absorption contour

$$\kappa_\ell^\nu = \kappa_\ell^c e^{-\frac{\pi(\nu-\bar{\nu}_c)^2}{\nu_D'^2}}, \tag{59}$$

where κ^c and $\bar{\nu}_c$ stand for the absorption coefficient at the line center of the ℓ-th component and the mean wave number of the whole system respectively, and ν_D' is the effective Doppler width defined by

$$\nu_D' = \sqrt{\pi}\,\nu_D = \frac{\sqrt{\pi}\,U\nu_c}{c}, \tag{60}$$

with

$$U = \sqrt{\frac{2kT}{M}}, \tag{61}$$

the most probable velocity of the fluorescing particles. In fact, one can write down the integrals of (55a, b) in terms of the curve of growth function (Cf. Landenberg, 1930; Struve and Elvey, 1934; Cowan and Dieke, 1948; Unsöld, 1955; Penner, 1959),

$$Z(y) = \frac{1}{\sqrt{\pi}}\int_{-\infty}^{\infty}\left[1 - e^{-y e^{-x^2}}\right] dx . \tag{62}$$

The result of the integration of (55a, b) over wave number is found to be

$$\mathcal{J}_\ell(t,+\mu) = \nu_D' \frac{\varpi_\ell}{k_\ell}\sum_{m=1}^{s} k_m f_m \frac{1}{1+\dfrac{\mu k_m}{\mu_0 k_\ell}} \times \left[Z\left(\left(\frac{k_m}{\mu_0}+\frac{k_\ell}{\mu}\right)t_0 - \frac{k_\ell}{\mu}t\right) - Z\left(\frac{k_m}{\mu_0}t\right)\right], \tag{63a}$$

and

$$\mathcal{J}_\ell(t,-\mu) = \nu_D' \frac{\varpi_\ell}{k_\ell}\sum_{m=1}^{s} k_m f_m \frac{1}{1-\dfrac{\mu k_m}{\mu_0 k_\ell}} \times \left[Z\left(\frac{k_\ell}{\mu}t\right) - Z\left(\frac{k_m}{\mu_0}t\right)\right]. \tag{63b}$$

In deriving (63a, b) it is assumed that f_m's are constant across the lines. This assumption will be usually fulfilled provided the incident flux is either continuum or a well-broadened line (Chamberlain and Sobouti, 1962). The right hand side of (63b) has a singularity when

$$\mu / k_\ell = \mu_0 / k_m .$$

In this case, one term of the singularity in (63b) should be replaced as

$$\lim \left\{ \frac{1}{1 - \frac{\mu k_m}{\mu_0 k_\ell}} \left[Z\left(\frac{k_\ell}{\mu} t\right) - Z\left(\frac{k_m}{\mu_0} t\right) \right] \right\} = \\ = \left[y \frac{dZ(y)}{dy} \right]_{y = \frac{k_\ell}{\mu} t} . \qquad (64)$$

IV. Line Intensity in the Atmosphere of Large Optical Thickness

1. <u>Resonant Intensity</u>

In the semi-infinite atmosphere, the terms of (44a, b) which increase infinitely at $t_0 = \infty$ should be excluded; so that b and $L_{-\alpha}$ are put equal to zero. Accordingly, the other sn constants a and $L_{+\alpha}$ can be determined from the boundary condition (42a) alone. It should be noted that these are no longer the functions of wave number, provided all the line components have the same absorption contour.

This kind of simplification for the semi-infinite atmosphere may be applicable also for the scattering from the atmosphere with large optical thickness. In this approximation, the "monochromatic" intensity is given by equations (44a, b).

Integration of (44a, b) over ν yields the line intensity

$$\mathcal{J}_\ell(t,+\mu) = \frac{\varpi_\ell}{k_\ell}\nu_D' \Bigg\{ Q\, Z\left(\frac{k_\ell}{\mu}(t_0-t)\right) +$$

$$+ \sum_{\alpha=1}^{sn-1} L_\alpha \frac{1}{1+\frac{\mu k_\alpha}{k_\ell}} \left[Z\left(\left(\frac{k_\ell}{\mu}+k_\alpha\right)t_0 - \frac{k_\ell}{\mu}t\right) - Z(k_\alpha t) \right] +$$

$$+ \sum_{m=1}^{s} k_m f_m \gamma_m \frac{1}{1+\frac{\mu k_m}{\mu_0 k_\ell}} \left[Z\left(\left(\frac{k_\ell}{\mu}+\frac{k_m}{\mu_0}\right)t_0 - \frac{k_\ell}{\mu}t\right) - Z\left(\frac{k_m}{\mu_0}t\right) \right] \Bigg\}, \qquad (65a)$$

and

$$\mathcal{J}_\ell(t,-\mu) = \frac{\varpi_\ell}{k_\ell}\nu_D' \Bigg\{ Q\, Z\left(\frac{k_\ell}{\mu}t\right) +$$

$$+ \sum_{\alpha=1}^{sn-1} L_\alpha \frac{1}{1-\frac{\mu k_\alpha}{k_\ell}} \left[Z\left(\frac{k_\ell}{\mu}t\right) - Z(k_\alpha t) \right] +$$

$$+ \sum_{m=1}^{s} k_m f_m \gamma_m \frac{1}{1-\frac{\mu k_m}{\mu_0 k_\ell}} \left[Z\left(\frac{k_\ell}{\mu}t\right) - Z\left(\frac{k_m}{\mu_0}t\right) \right] \Bigg\}. \qquad (65b)$$

2. Fluorescent Intensity

The fluorescent intensity can be deduced as the special case of (65a, b) where $k_\ell = 0$. Since $Q \equiv 0$, if any fluorescent line exists, it is given by

$$\mathcal{J}_\ell(t,+\mu) = \varpi_\ell \nu_D' \frac{1}{\mu} \Bigg\{ \sum_{\alpha=1}^{sn} \frac{L_\alpha}{k_\alpha} \left[Z(k_\alpha t_0) - Z(k_\alpha t) \right] +$$

$$+ \sum_{m=1}^{s} f_m \gamma_m \mu_0 \left[Z\left(\frac{k_m}{\mu_0}t_0\right) - Z\left(\frac{k_m}{\mu_0}t\right) \right] \Bigg\}, \qquad (66a)$$

and

$$\mathcal{J}_\ell(t,-\mu) = \varpi_\ell \nu_D' \frac{1}{\mu}\left\{ \sum_{\alpha=1}^{sn} \frac{L_\alpha}{k_\alpha} Z_1(k_\alpha t) + \right.$$

$$\left. + \sum_{m=1}^{s} f_m \gamma_m \mu_0 \, Z_1\left(\frac{k_m}{\mu_0} t\right) \right\}, \qquad \text{(Fluorescent lines)} \tag{66b}$$

3. <u>Line Intensity at Large Optical Depth</u>

When both the conditions, t_0, $t \gg 1$, and $t_0 - t \gg 1$ are satisfied, all the terms except those with Q vanish in (65a, b) because the Z-function in this case is apparently constant, being independent of its argument. Therefore, we have for large extent of μ,

$$\mathcal{J}_\ell(t,+\mu) \approx \mathcal{J}_\ell(t,-\mu) \approx \frac{\varpi_\ell}{k_\ell} \nu_D' Q \, Z_0, \tag{67}$$

where Zo stands for the average value of Z(y) and actually may be set equal to $Z(k_\ell t_0)$. This result means that in the conservative case, the radiation field tends to be isotropic as the depth increases and the relative intensity approaches $\frac{\varpi_\ell}{k_\ell}$. For non-conservative cases ($\sum_{m=1}^{s} \varpi_m < 1$; or $k_r = 0$ ($r \in s$) though $\sum_{m=1}^{s} \varpi_m \equiv 1$), Q is equal to zero, so that the "resonant" intensities vanish at large optical depth whereas the fluorescent intensities have finite values as

$$\mathcal{J}_\ell(t,+\mu) \longrightarrow 0, \qquad \text{(Resonant Lines)}, \tag{68a}$$

and

$$\mathcal{J}_\ell(t,-\mu) \rightarrow \varpi_\ell \nu_0' \frac{1}{\mu}\left\{ \sum_{\alpha=1}^{sn} \frac{L_\alpha}{k_\alpha} + \sum_{m=1}^{s} f_m \gamma_m \mu_0 \right\} Z_0, \tag{68b}$$

(Fluorescent Lines, t_0, $t \gg 1$).

where Zo again stands for the average value of $Z(k_\alpha\ t)$ and $Z_1(\ km/\mu_o\ t)$.

These results show that at large optical depth the radiation is entirely pumped into the fluorescent lines.

V. Effect of Cascade Transitions

1. Preliminary Considerations

As mentioned in the introductory part of this paper, sometimes it happens that the upper state of a coupled fluorescent system coincides with the final state of other fluorescent system (Brandt, 1959). In this case, the atmosphere has an active source inside of it. Let us discuss the practical treatment of such an effect on the fluorescent system. The grand source function (22) is then replaced by

$$J(t) = J^s_{(t)} + J^e_{(t)} + J^i_{(t)}, \tag{69}$$

where $J^s(t)$ and $J^e(t)$ are defined as before (Eq.'s 13 and 14), and $J^i(t)$ stands for the active source function inside of the atmosphere. However, because of the linearity of Eq. (20) the complete solution will be the sum of the two solutions which are solved for the external and internal sources respectively. Therefore, we may put $J^e(t) \equiv 0$ initially. First we look for the solution for an active source of the form

$$J^i(k,t) = g(k)\, e^{-kt}, \qquad (0 \le k < +\infty). \tag{70}$$

The radiation field, due to this grand source function, will be governed by the linear differential equations,

$$-\mu_i \frac{dI_{\ell i}}{dt} + k_\ell I_{\ell i} = \frac{\varpi_\ell}{2} \sum_{m=1}^{s} k_m \sum_{j=-n}^{+n} a_j I_{mj} + \varpi_\ell g(k) e^{-kt}. \tag{71}$$

The particular solutions for this set of differential equations can be obtained in quite an analogous way with that adopted in Sec. I. They are found to be

$$I_{\ell i}(k) = \frac{\varpi_\ell}{k_\ell} \frac{\gamma(k)}{1 + \frac{\mu_i k}{k_\ell}} e^{-kt}, \tag{72}$$

with

$$\gamma(k) = \frac{1}{1 - \sum_{m=1}^{s} \varpi_m \sum_{j=-n}^{+n} \frac{a_j}{1 - \left(\frac{\mu_j k}{k_m}\right)^2}}. \tag{73}$$

The solutions of the associated homogeneous system are again given by Eq. (31), but the constants b(k), Q(k) and L_α (k)'s must satisfy the boundary conditions, corresponding to (42a,b),

$$[I_{\ell -i}]_{t=0} = 0;$$

$$-b(k) \frac{\mu_i}{k_\ell} + Q(k) + \sum_{\alpha=-sn+1}^{sn-1} \frac{L_\alpha(k)}{1 - \frac{\mu_i k_\alpha}{k_\ell}} + \frac{\gamma(k)}{1 - \frac{\mu_i k}{k_\ell}} = 0, \tag{74a}$$

and

$$[I_{\ell +i}]_{t=0} = 0;$$

$$b(k)\left(t_0 + \frac{\mu_i}{k_\ell}\right) + Q(k) + \sum_{\alpha=-sn+1}^{sn-1} \frac{L_\alpha(k) e^{-k_\alpha t_0}}{1 + \frac{\mu_i k_\alpha}{k_\ell}} + \sum \frac{\gamma(k) e^{-kt_0}}{1 + \frac{\mu_i k}{k_\ell}} = 0, \tag{74b}$$

$$(i = +1, +2, \cdots, +n, \ \ell = 1, 2, \cdots, s).$$

Then, the source function and the specific intensities corresponding to (43) and (44a, b) are respectively,

$$J_\ell(t) = \varpi_\ell\, J(t) = \varpi_\ell \left[b(k)t + Q(k) + \sum_{\alpha=-sn+1}^{sn-1} L_\alpha(k)\, e^{-k_\alpha t} + g(k)\gamma(k)\, e^{-kt} \right], \tag{75}$$

$$\begin{aligned} I_\ell(k;\, t,+\mu) = \frac{\varpi_\ell}{k_\ell} \Bigg\{ & b(k)\left[\left(\frac{\mu}{k_\ell}+t\right) - \left(\frac{\mu}{k_\ell}+t_0\right) e^{-\frac{k_\ell(t_0-t)}{\mu}} \right] + \\ & + Q(k)\left[1 - e^{-\frac{k_\ell(t_0-t)}{\mu}} \right] + \\ & + \sum_{\alpha=-sn+1}^{sn-1} L_\alpha(k)\, \frac{1}{1+\frac{\mu k_\alpha}{k_\ell}} \left[e^{-k_\alpha t} - e^{-\left(\frac{k_\ell}{\mu}+k_\alpha\right)t_0 + \frac{k_\ell}{\mu}t} \right] + \\ & + g(k)\gamma(k)\, \frac{1}{1+\frac{\mu k}{k_\ell}} \left[e^{-kt} - e^{-\left(\frac{k_\ell}{\mu}+k\right)t_0 + \frac{k_\ell}{\mu_0}t} \right] \Bigg\}. \end{aligned} \tag{76a}$$

and

$$\begin{aligned} I_\ell(k;\, t,-\mu) = \frac{\varpi_\ell}{k_\ell} \Bigg\{ & b(k)\left[t - \frac{\mu}{k_\ell}\left(1 - e^{-\frac{k_\ell}{\mu}t}\right) \right] + \\ & + Q(k)\left[1 - e^{-\frac{k_\ell}{\mu}t} \right] + \\ & + \sum_{\alpha=-sn+1}^{sn-1} L_\alpha(k)\, \frac{1}{1-\frac{\mu k_\alpha}{k_\ell}} \left[e^{-k_\alpha t} - e^{-\frac{k_\ell}{\mu}t} \right] + \\ & + g(k)\gamma(k)\, \frac{1}{1-\frac{\mu k}{k_\ell}} \left[e^{-kt} - e^{-\frac{k_\ell}{\mu}t} \right] \Bigg\}. \end{aligned} \tag{76b}$$

Now it may be seen that so far as mathematical procedures are concerned, there is no difficulty in extending the theory to an arbitrary distribution of an active source function, if it is given a priori, since any distribution of source, J(t) can be written as

$$J(t) = \int_0^\infty g(k)\, e^{-kt}\, dk \,, \tag{77}$$

where g(k) is the Laplace transformation of J(t), i.e.,

$$g(k) = \frac{1}{2\pi i} \int_{a-i\infty}^{a+i\infty} e^{kt} J(t)\, dt \,. \tag{78}$$

In fact, the solutions for this source distribution will be obtained by integrating the solution for g(k) over k.

2. Approximation in the Treatment of Cascade Transition

Suppose the coupling resonant and fluorescent system $X_m \rightleftarrows A_p$, (m = 1, 2, ..., s) whose upper state A_p is also populated by the other fluorescent systems, $X_n \rightleftarrows B_q \rightarrow A_p$, (n = 1, 2, ..., s; q = 1, 2, ..., r), which are excited by the incident fluxes f_{qn}, (n = 1, 2, ..., s; q = 1, 2, ..., r), (Cf. Fig. 3). If there is no reaction from the former system to to the latter, the radiative quantities for the latter systems may be approximated in the treatment as developed in Sec. I. The source functions for transition q→p of each system may be written as

$$J_{qp}(t_q) = \varpi_{qp} \left[\sum_\rho L_{q\rho}\, e^{-k_{q\rho} t_q} + \sum_n k_{qn} f_{qn} \gamma_{qn}\, e^{-\frac{k_{qn}}{\mu_0} t_q} \right], \tag{79}$$

where every quantity is specified by its corresponding transition. Then, the source function for the ℓ-th component of the $X_\ell \rightarrow A_p$ system due to this cascading is

$$J_{p\ell}^{i}(t_p) = \varpi_{p\ell} \sum_{g} \lambda_{gp} \varpi_{gp} \times$$

$$\times \left[\sum_{\beta} L_{g\beta} e^{-\lambda_{gp} k_{g\beta} t_p} + \sum_{n} k_{gn} f_{gn} \gamma_{gn} e^{-\frac{\lambda_{gp} k_{gn}}{\mu_0} t_p} \right], \tag{80}$$

where

$$\lambda_{gp} = \frac{t_g}{t_p} = \frac{\sum_n \varkappa_{gn}}{\sum_m \varkappa_{pm}}, \tag{81}$$

is the ratio of optical thickness of cascading and cascaded systems. As stated in Sec. V. - 1, there would be no necessity to consider the population of A_p from X_m by direct excitation due to external sources, since the final solutions will be obtained by the simple summation of the two independent solutions, one for the external source and the other for the cascade population.

Therefore, one can assume that

$$f_{pm} \equiv 0, \qquad (m = 1, 2, \ldots, s) \tag{82}$$

Under this assumption, the equations of radiative transfer in the n-th approximation are

$$-\mu_i \frac{dI_{pl,i}}{dt_p} + k_{pl} I_{pl,i}$$
$$= \frac{\varpi_{pl}}{2} \sum_m k_{pm} \sum_j a_j I_{pm,j} + \varpi_{pl} \left\{ \sum_g \lambda_{gp} \varpi_{gp} \times \right.$$
$$\left. \times \left[\sum_\beta L_{g\beta} e^{-\lambda_{gp} k_{g\beta} t_p} + \sum_n k_{gn} f_{gn} \gamma_{gn} e^{-\frac{\lambda_{gp} k_{gn}}{\mu_0} t_p} \right] \right\}. \tag{83}$$

By making use of the general treatment of the previous paragraph, they are solved as

$$I_{pl,i} = \frac{\varpi_{pl}}{k_{pl}} \left\{ b_p \left(t + \frac{\mu_i}{k_{pl}} \right) + Q_p + \sum_\alpha \frac{L_{p\alpha} e^{-k_{p\alpha} t_p}}{1 + \frac{\mu_i k_{p\alpha}}{k_{pl}}} + \right.$$
$$+ \sum_g \lambda_{gp} \varpi_{gp} \sum_\beta L_{g\beta} \Gamma_{g\beta} \frac{e^{-\lambda_{gp} k_{g\beta} t_p}}{1 + \frac{\mu_i \lambda_{gp} k_{g\beta}}{k_{pl}}} +$$
$$\left. + \sum_g \lambda_{gp} \bar{\omega}_{gp} \sum_n k_{gn} f_{gn} \gamma_{gn} \Gamma_{gn} \frac{e^{-\frac{\lambda_{gn} k_{gn}}{\mu_0} t_p}}{1 + \frac{\mu_i \lambda_{gp} k_{gn}}{\mu_0 k_{pl}}} \right\}, \tag{84}$$

where

$$\Gamma_{g\beta} = \frac{1}{1 - \sum_l \varpi_{pl} \sum_{j=1}^{+n} \frac{a_j}{1 - \left(\frac{\mu_j \lambda_{gp} k_{g\beta}}{k_{pl}} \right)^2}}, \tag{85}$$

and

$$\Gamma_{\gamma n} = \frac{1}{1 - \sum_{\ell} \varpi_{p\ell} \sum_{j=1}^{+n} \frac{a_j}{1 - \left(\frac{\mu_j \lambda_{\gamma p} k_{\gamma n}}{\mu_0 k_{p\ell}} \right)^2}} \tag{86}$$

The constants in (84) b_p Q_p and $L_{p\alpha}$ can be determined from the boundary conditions

$$[\, I_{p\ell,-i} \,]_{t_p = 0} = 0 \,, \tag{87a}$$

and

$$[\, I_{p\ell,+i} \,]_{t_p = t_{p0}} = 0 \,. \tag{87b}$$

Whence, the average intensity $J_{p\ell}(t_p)$ and the specific intensity may be deduced respectively to be

$$J_{p\ell}(t_p) = \varpi_{p\ell} \Big\{ b_p t_p + Q_p + \sum_{\alpha} L_{p\alpha} \, e^{-k_{p\alpha} t_p} + \\ + \sum_{\gamma} \lambda_{\gamma p} \varpi_{\gamma p} \Big[\sum_{\beta} L_{\gamma\beta} \Gamma_{\gamma\beta} \, e^{-\lambda_{\gamma p} k_{\gamma\beta} t_p} + \\ + \sum k_{\gamma n} f_{\gamma n} \gamma_{\gamma n} \Gamma_{\gamma n} \, e^{-\frac{\lambda_{\gamma p} k_{\gamma n} t_p}{\mu_0}} \Big] \Big\} , \tag{88}$$

$$I_{p\ell}(t_p,+\mu) =$$

$$= \frac{\varpi_{p\ell}}{k_{p\ell}} \Bigg[b_p \Big[\Big(\frac{\mu}{k_{p\ell}} + t_p\Big) - \Big(\frac{\mu}{k_{p\ell}} + t_{po}\Big) e^{-\frac{k_{p\ell}(t_{po}-t_p)}{\mu}} \Big] +$$

$$+ Q_p \Big[1 - e^{-\frac{k_{p\ell}(t_{po}-t_p)}{\mu}} \Big] +$$

$$+ \sum_\alpha L_{p\alpha} \frac{1}{1+\frac{\mu k_{p\alpha}}{k_{p\ell}}} \Big[e^{-k_{p\alpha} t_p} - e^{-\left(\frac{k_{p\ell}}{\mu}+k_{p\alpha}\right)t_{po}+\frac{k_{p\ell}}{\mu}t_p} \Big] +$$

$$+ \sum_g \lambda_{gp} \varpi_{gp} \Big\{ \sum_\beta L_{g\beta} \Gamma_{g\beta} \frac{1}{1+\frac{\mu\lambda_{gp}k_{g\beta}}{k_{p\ell}}} \times$$

$$\times \Big[e^{-\lambda_{gp}k_{g\beta}t_p} - e^{-\left(\frac{k_{p\ell}}{\mu}+\lambda_{gp}k_{g\beta}\right)t_{po}+\frac{k_{p\ell}}{\mu}t_p} \Big] +$$

$$+ \sum_n k_{gn} f_{gn} \gamma_{gn} \Gamma_{gn} \frac{1}{1+\frac{\mu\lambda_{gp}k_{gn}}{\mu_o k_{p\ell}}} \times$$

$$\times \Big[e^{-\frac{\lambda_{gp}k_{gn}}{\mu_o}t_p} - e^{-\left(\frac{k_{p\ell}}{\mu}+\frac{\lambda_{gp}k_{gn}}{\mu_o}\right)t_{po}+\frac{k_{p\ell}}{\mu}t_p} \Big] \Big\} \Bigg] . \tag{89a}$$

and

$$I_{p\ell}(t_p,-\mu) =$$

$$= \frac{\varpi_{p\ell}}{k_{p\ell}} \Bigg[b_p \Big[t_p - \frac{\mu}{k_{p\ell}}\Big(1 - e^{-\frac{k_{p\ell}}{\mu}t_p}\Big) \Big] +$$

$$+ Q \Big[1 - e^{-\frac{k_{p\ell}}{\mu}t_p} \Big] +$$

$$+ \sum_{\alpha=-Sn+1}^{Sn-1} L_{p\alpha} \frac{1}{1-\frac{\mu k_{p\alpha}}{k_{p\ell}}} \Big[e^{-k_\alpha t_p} - e^{-\frac{k_{p\ell}}{\mu}t_p} \Big] +$$

$$+ \sum_g \lambda_{gp} \varpi_{gp} \Big\{ \sum_\beta L_{g\beta} \Gamma_{g\beta} \frac{1}{1-\frac{\mu\lambda_{gp}k_{g\beta}}{k_{p\ell}}} \times \Big[e^{-\lambda_{gp}k_{g\beta}t_p} - e^{-\frac{k_{p\ell}}{\mu}t_p} \Big] +$$

$$+ \sum k_{gn} f_{gn} \gamma_{gn} \Gamma_{gn} \frac{1}{1-\frac{\mu\lambda_{gp}k_{gn}}{\mu_o k_{p\ell}}} \times \Big[e^{-\frac{\lambda_{gp}k_{gn}}{\mu_o}t_p} - e^{-\frac{k_{p\ell}}{\mu}t_p} \Big] \Big\} \Bigg] . \tag{89b}$$

VI. Numerical Example - (OI)1302-4-6A Dayglow

An example of a coupled resonance system is found in the ultraviolet dayglow of atomic oxygen, (OI)1302-4-6A. This triplet emission is believed to originate from two different sources, namely

1. Direct resonance of solar (OI)1302-4-6A emission,

$$O(2p^4\ {}^3P_{2,1,0}) + h\nu\ (1302\text{-}4\text{-}6) \rightleftharpoons O(2p^3 3s\ {}^3S_1); \tag{90}$$

2. Cascade transition initiated by solar Lyman-β emission,

$$O(2p^4\ {}^3P_2) + h\nu\ (L_\beta) \rightarrow O(2p^3 3d\ {}^3D_{3,2,1}) \rightarrow$$
$$\rightarrow \begin{cases} O(2p^3 3p\ {}^3P_{2,1,0}) + h\nu\ (11260), \\ O(2p^4\ {}^3P_{2,1,0}) + h\nu\ (1026\text{-}7\text{-}8) \end{cases} \tag{91}$$

$$O(2p^3 3p\ {}^3P_{2,1,0}) \rightarrow O(2p^3 3s\ {}^3S_1) + h\nu\ (8446) \rightarrow$$
$$\rightarrow\ O(2p^4\ {}^3P_{2,1,0}) + h\nu\ (1302\text{-}4\text{-}6)\ . \tag{92}$$

The latter mechanism was originally suggested by Bowen (1947) for astrophysical interest and was studied by Brandt (1959) in connection with the (OI)11260A- and 8446A- dayglow. Since solar (OI)1302-4-6 and Lyman-β emissions have intensities of the same order of magnitude as shown in Table I, both the mechanisms (1) and (2) will contribute to the formation of (OI)1302-4-6A dayglow. Albedos, populations in the ground states specific absorption coefficients and optical depths for systems (1) and (2) can be calculated with the aid of the relations,

$$\kappa^c = \frac{\pi e^2}{mc^2}\,\frac{f}{\nu_D'}\,\frac{n^*}{n}\,, \tag{93}$$

and

$$t = \sum \kappa^c \cdot N(0,z), \tag{94}$$

with

$$N(0,z) = \int_z^{\infty} n(0,z)\,dz, \tag{95}$$

$$\nu_D' = \sqrt{\pi}\,\nu_D = \sqrt{\frac{\pi k T}{2 M c^2}}\,\bar{\nu}_c , \tag{96}$$

and $\frac{n^*}{n}$ the relative populations in each of the ground states. The albedos are calculated from the relation

$$\varpi_\ell = \frac{g_\ell f_\ell}{\sum g_m f_m}, \quad (\ell = 1, 2, 3),$$

where $f_1 = f_2 = f_3$ and g is the degeneracy of each level. The oscillator strengths are tabulated in Table III.

The optical depths for the systems (1) and (2) are then computed from

$$\left.\begin{aligned} t &= \sum_J \tau^c({}^3P_J - {}^3S_1) = 5.334 \times 10^{-14}\, N(0,z), \\ t(L_\beta) &= \sum_J \tau^c({}^3P_2 - {}^3D_J) = 1.064 \times 10^{-14}\, N(0,z), \end{aligned}\right\} \quad (T = 800\,°K). \tag{97}$$

The absorption due to molecular oxygen will prevent the Lyman-β and (OI)1302-4-6A radiations from penetrating lower than the 100 km level. The values of optical depths for these radiations are respectively $1.97 \times 10^{+4}$ and $9.87 \times 10^{+4}$ at 100 km. Therefore, the earth's atmosphere is optically thick for these radiations (Table IV).

The line intensities of (OI)1302-4-6 A dayglow are then obtained by means of direct application of the approximation procedures developed in sections IV and V. Figures 4-a, b, c and d show the line intensities of 1306A, 1304A and 1302A components for unit inputs of solar 1306A, 1304A, 1302A and 1026(L_β) A radiations respectively in the conservative case as observed looking up (μ= -1.000) when the solar radiation is straight down. These curves are obtained for the first approximation in the source function (n=1). Because of the linearity of the transfer equations with respect to incident radiations, the actual altitude distribution will be a linear combination of the curves, the combination constants being determined by the relative amounts of incident fluxes.

VII. Discussions

At the top and bottom of the scattering atmosphere, the intensities deduced from the quadrature approximation, (44a,b) will approach the exact values which the rigorous theory provides in terms of generalized H- or X- and Y- functions. The proof is rather complex but fundamental (Cf. R. T. III 26-5 and VIII 59-1). For example, Sobouti's generalized H-function can be derived in the n-th approximation as

$$\mathcal{H}(x) = \frac{(k_1 k_2 \cdots k_s)^n}{(\mu_1 \mu_2 \cdots \mu_n)^s} \frac{\prod_{m=1}^{s}\left(\prod_{j=1}^{n}\left(x + \frac{\mu_j}{k_m}\right)\right)}{\prod_{\alpha=1}^{sn}(1 + k_\alpha x)} . \tag{98}$$

Then, diffusely scattered intensities emerging from the top of semi-infinite atmosphere are given by

$$I_\ell(0,+\mu) = \frac{1}{\mu}\sum_{m=1}^{s} S_{\ell m}(\mu,\mu_0)\, f_m , \tag{99}$$

where $S_{\ell m}(\mu,\mu_0)$ is one of the elements of a scattering matrix defined by

$$S_{\ell m}(\mu,\mu_0) = \varpi_\ell k_m \frac{\mathcal{H}(\mu/k_\ell)\,\mathcal{H}(\mu_0/k_m)}{\frac{k_\ell}{\mu} + \frac{k_m}{\mu_0}} . \tag{100}$$

Rational representation of generalized X- and Y- functions will be derived in quite an analogous way to R. T. VIII 59-1.

The foregoing discussions on coupled radiative transfer problems are based on the assumption that the scattering by fluorescent particles takes place coherently at the same wavelength with the incident radiation. The Doppler incoherency due to motion of fluorescent particles, as estimated by Henyey (1940), may give a modification to the theory of radiative transfer especially in an optically thick atmosphere. It would be worth noticing in this connection that the theory of coupling in sections I and IV can be extended to the case of Doppler incoherent effect where the wavelength shift in the scattering process can be regarded as a "continuous" coupling of radiation of the incident wavelength to that of the emergent wavelengths. In order to see such an effect readily, one can consider the line feature in the Doppler core. As shown by Jeffries and White (1960), spectral distribution of scattered radiation is approximately Gaussian in the Doppler core, being independent of the incident frequency. In this approximation, the isotropic equation of transfer for a single component system is written as

$$-\mu \frac{dI^{x}(t,\mu)}{dt} + e^{-x^2} I^{x}(t,\mu) = = \frac{\varpi e^{-x^2}}{\sqrt{\pi}} \frac{1}{2} \int_{-1}^{+1} d\mu' \int_{-\infty}^{\infty} e^{-x'^2} I^{x'}(t,\mu')\, dx' + + \frac{\varpi e^{-x^2}}{\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-x'^2} f\, e^{-\frac{t e^{-x'^2}}{\mu_0}}\, dx'. \tag{101}$$

Being analogous to the transition from Eq. (8) to Eq. (20) in the radiative transfer equations, Eq. (101) can be substituted for by an equivalent set of linear equations of intensities at finite equi-distant points of frequencies, $x = x_1, x_2, x_3, \ldots, x_q$, namely,

$$-\mu_i \frac{dI_{pi}}{dt} + e^{-x_p^2} I_{pi} = \frac{\varpi \Delta x\, e^{-x_p^2}}{\sqrt{\pi}} \frac{1}{2} \sum_q e^{-x_q^2} \sum_{j=-n}^{+n} a_j I_{qj} + $$
$$+ \frac{\varpi \Delta x\, e^{-x_p^2}}{\sqrt{\pi}} \sum_q e^{-x_q^2} f_q\, e^{-\frac{e^{-x_q^2}}{\mu_0} t} .$$

(102)

This equation has exactly the same form as Eq. (20), if one compares the symbols and constants of Eq.'s (20) and (102) in the following scheme:

$$\left.\begin{array}{l} p \rightleftarrows l, \quad q \rightleftarrows m, \\ e^{-x_p^2} \rightleftarrows k_l, \quad e^{-x_q^2} \rightleftarrows k_m, \\ \dfrac{\varpi \Delta x\, e^{-x_p^2}}{\sqrt{\pi}} \rightleftarrows \varpi_l, \\ f_q \rightleftarrows f_m . \end{array}\right\}$$

(103)

while the role of i and j being unchanged. Therefore, the line profile can be estimated to an arbitrary degree of approximation with the same procedure as Sec. 1. Especially in the conservative case, since

$$\sum_p \frac{\varpi \Delta x\, e^{-x_p^2}}{\sqrt{\pi}} \equiv 1, \tag{104}$$

Eq. (102) has a set of particular integrals,

$$I_{pi} = \frac{\varpi \Delta x}{\sqrt{\pi}} \sum_{-\gamma n}^{+\gamma n} L_\alpha \frac{1}{1 + \frac{\mu_i k_\alpha}{e^{-x_p^2}}} e^{-k_\alpha t} , \tag{105a}$$

and

$$I_{p,i} = Q = const. \tag{105b}$$

Since the integral (105a) tends to zero at large optical depths, we have for large values of t,

$$I^x(t,+\mu) = Q\left[1 - e^{-\frac{e^{-x^2}}{\mu}(t_0-t)}\right], \tag{106a}$$

and

$$I^x(t,-\mu) = Q\left[1 - e^{-\frac{e^{-x^2}}{\mu}t}\right]. \tag{106b}$$

Consequently, the spectral line will become flat and isotropic at large optical depths. This effect is similar to the broadening of stellar absorption lines by electron scattering (Münch, 1948). Then, the dayglow emissions observed at large optical depths will have extended wings and these wings may form a false continuum to the emission lines. In this case, the measured line intensity may deviate from the theoretical line intensity.

VIII. Acknowledgement

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?

TABLE I

SOLAR RADIATION DATA FOR (OI)1302-4-6 DAYGLOW

		L-β	(OI)			
Wavelength (A)		1026	1302	1304	1306	(OI) Total
Total flux of solar line in erg/cm^2/sec	Tousey[1] Johnson[2]	0.06 0.2	0.013 0.02	0.020 0.03	0.025 0.04	0.058 0.09
Total flux of solar line in 10^9 p/cm^2/sec	Tousey[1] Johnson[2]	3.10 10.33	0.86 1.33	1.35 2.04	1.68 2.54	3.81 5.91
πF^c in 10^9/cm^2/sec/wave number[3]	Tousey[1] Johnson[2]	0.0236 0.109	0.0147 0.0226	0.0227 0.0348	0.0283 0.0434	0.0659 0.101
πF^c $_D$' in 10^9 p/cm^2/sec[4]	Tousey[1] Johnson[2]	0.0170 0.0568	0.00605 0.00925	0.00930 0.0143	0.0117 0.0178	0.0268 0.0415

1. Detwiler, C. R., Garret, D. L., Purcell, J. D., and Tousey, R., 1961.

2. Johnson, F. S., 1961

3. The effective width of the line is assumed to be 1Å.

4. The Doppler widths of L-β and (OI) lines are assumed to be 0.523 cm^{-1} and 0.412 cm^{-1} respectively, corresponding to T = 800°K.

TABLE II

RADIATIVE TRANSFER ELEMENTS OF (OI) 1302-4-6A RESONANCE SYSTEM

	Transition	Wavelength	g		n*/n	k
1	$^3S_1$ - ^3P_o	1306.03	1	0.1111	0.0834	0.0834
2	$^3S_1$ - $^3P_1$	1304.87	3	0.3333	0.2845	0.2845
3	$^3S_1$ - $^3P_2$	1302.42	5	0.5556	0.6321	0.6321

TABLE III

OSCILLATOR STRENGTHS

Transition	f	Reference
$2p^4\ ^3P_{2,1,0}$ - $2p^33s\ ^3S_1$	0.0299	Kelly, 1963
$2p^4\ ^3P_2$ - $2p^33d\ ^3D_3$	0.01	Omholt, 1956
$2p^4\ ^3P_2$ - $2p^33d\ ^3D_2$	0.0017	
$2p^4\ ^3P_2$ - $2p^33d\ ^3D_1$	0.0001	

TABLE IV

MODEL ATMOSPHERE FOR (OI)1302-4-6 RESONANCE SYSTEM

AND OPTICAL DEPTHS

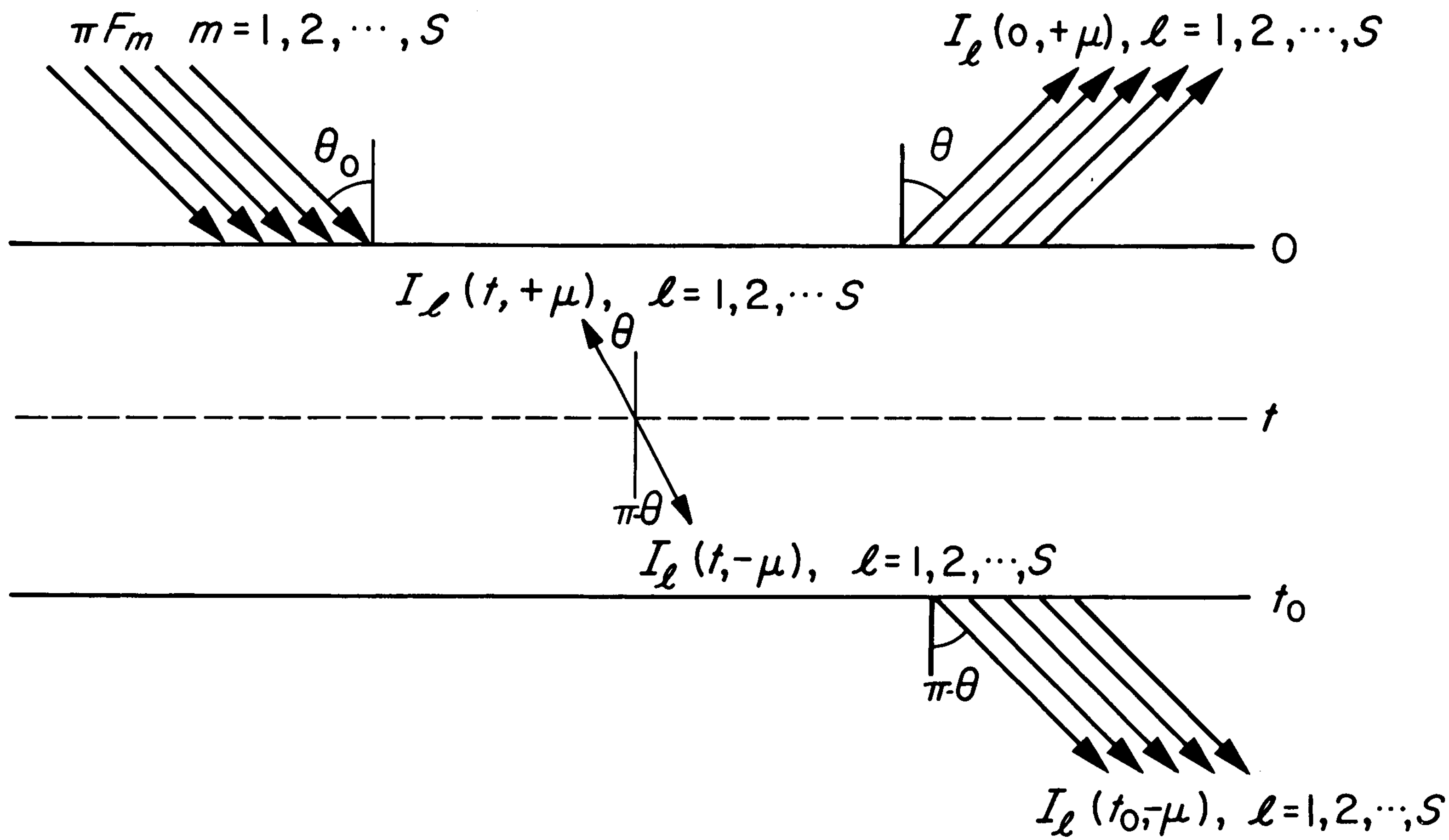
Height (km)	n(O) (cm^{-3})	N(O) (cm^{-2})	T (°K)	t Optical Depth	t(Lβ) Optical Depth
100	1.70+12	1.85+18	200	9.87+4	1.97+4
120	1.80+11	3.74+17	380	1.99+4	4.00+3
140	5.60+10	1.72+17	560	9.17+3	1.83+3
160	2.52+10	9.78+16	700	5.22+3	1.04+3
180	1.34+10	6.15+16	824	3.28+3	6.56+2
200	7.80+9	4.12+16	941	2.20+3	4.40+2
300	1.17+9	9.77+15	1445	5.21+2	1.04+2
500	max 1.18+8	5.88+15	1500	3.14+2	6.27+1
	min 1.00+8	2.66+15	1000	1.42+2	2.83+1
1000	max 5.82+5	3.79+13	1500	2.03+0	4.05-1
	min 4.73+3	1.82+11	1000	9.73+3	1.94+3

References:

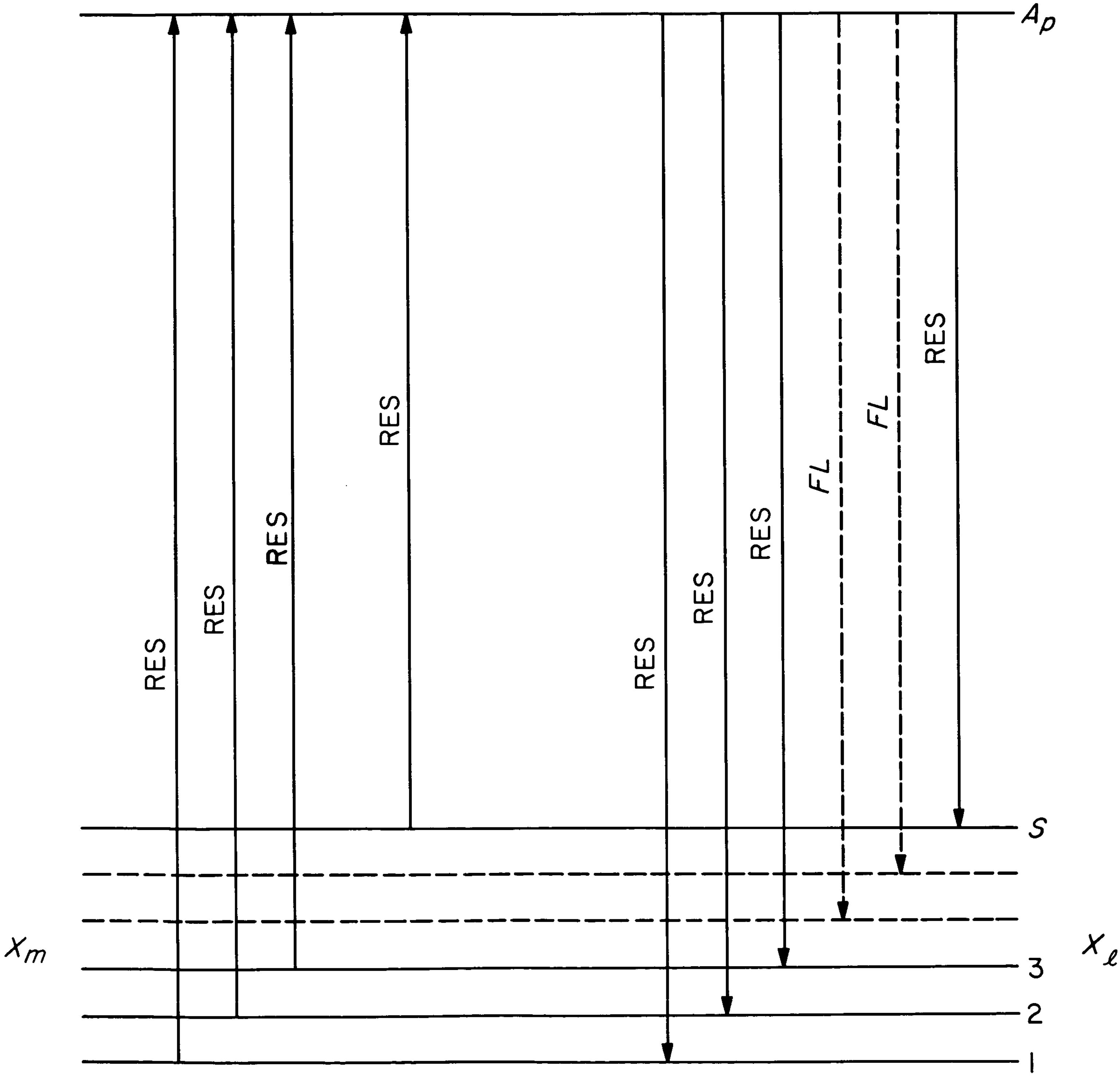
Below 300 km: Chamberlain, 1961

Above 500 km: Johnson, 1961

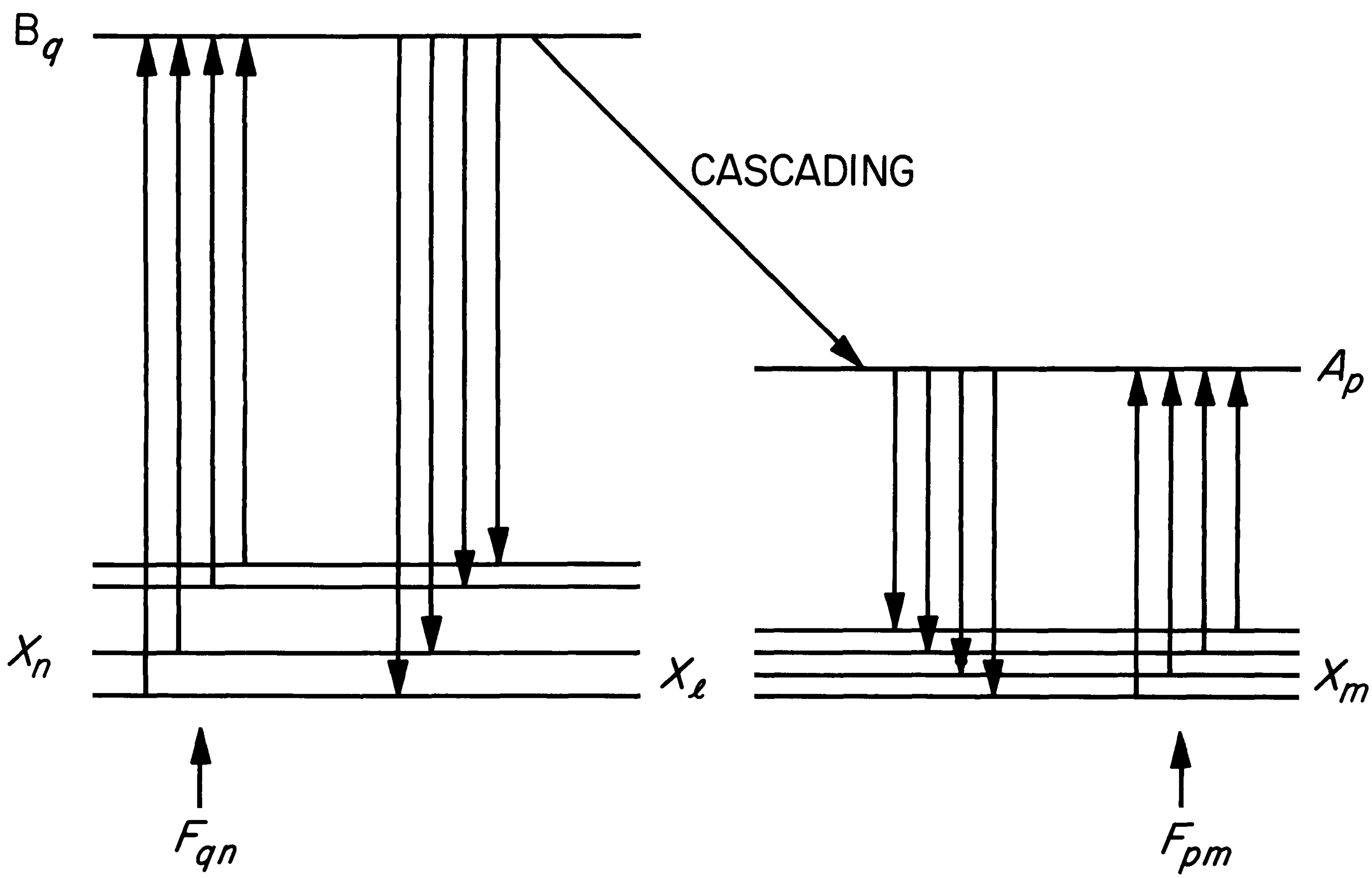
LIST OF FIGURES



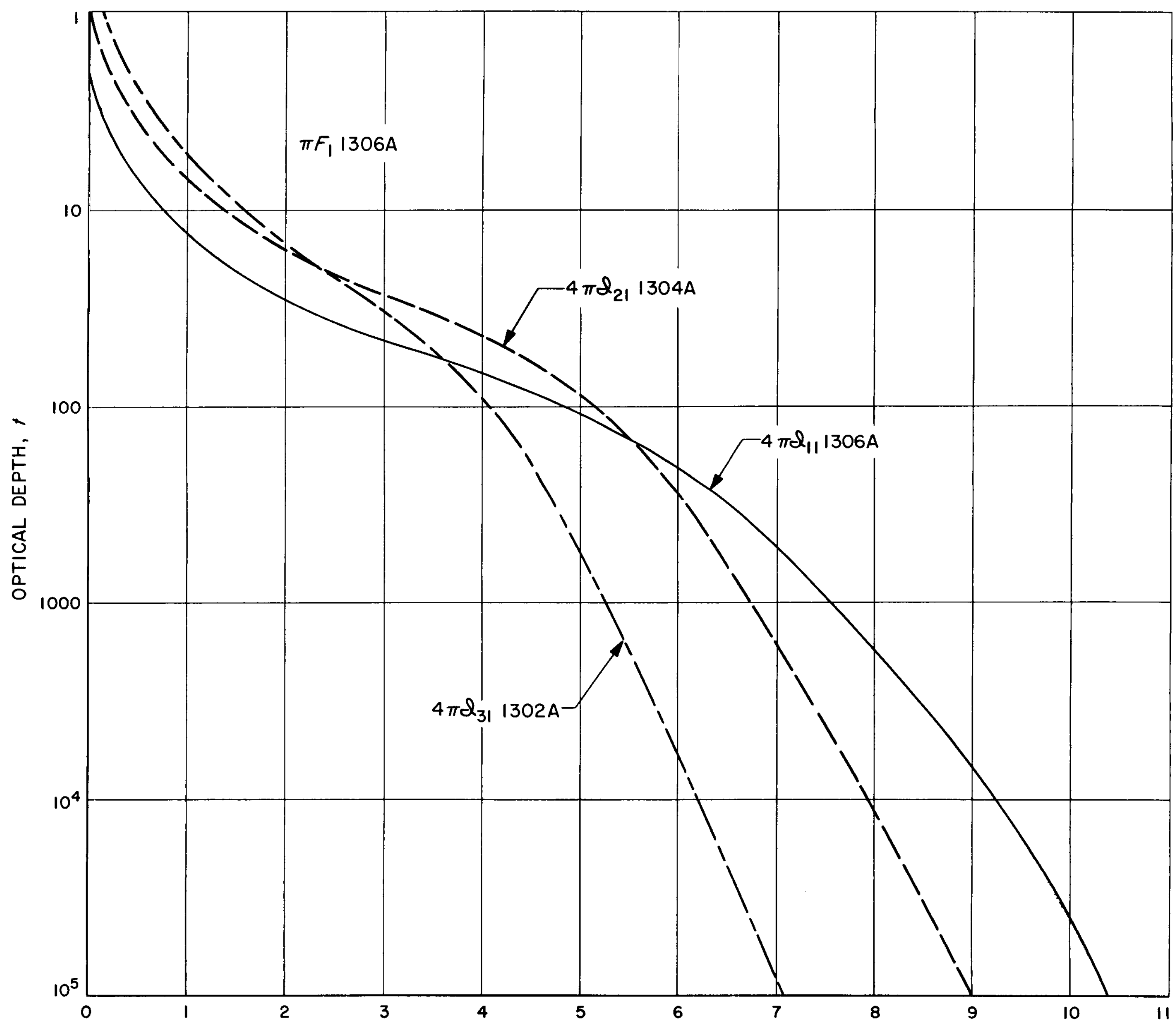
πF_m $m=1,2,\cdots,S$
θ_0
$I_\ell(0,+\mu)$, $\ell=1,2,\cdots,S$
θ
0
$I_\ell(t,+\mu)$, $\ell=1,2,\cdots S$
θ
t
$\pi\text{-}\theta$
$I_\ell(t,-\mu)$, $\ell=1,2,\cdots,S$
t_0
$\pi\text{-}\theta$
$I_\ell(t_0,-\mu)$, $\ell=1,2,\cdots,S$



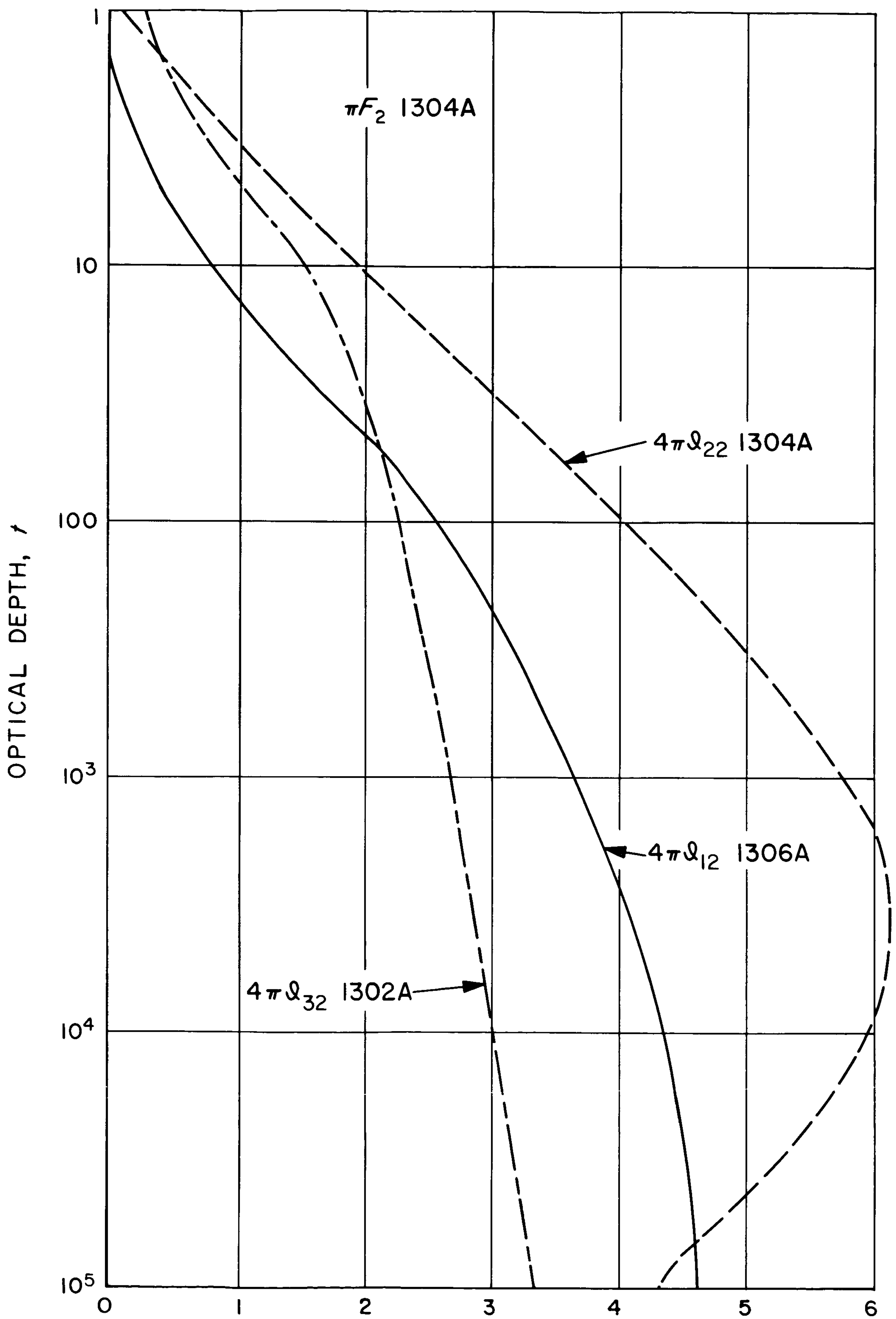
A_p
RES
RES
RES
RES
RES
RES
RES
FL
FL
RES
S
X_m
X_ℓ
3
2
1



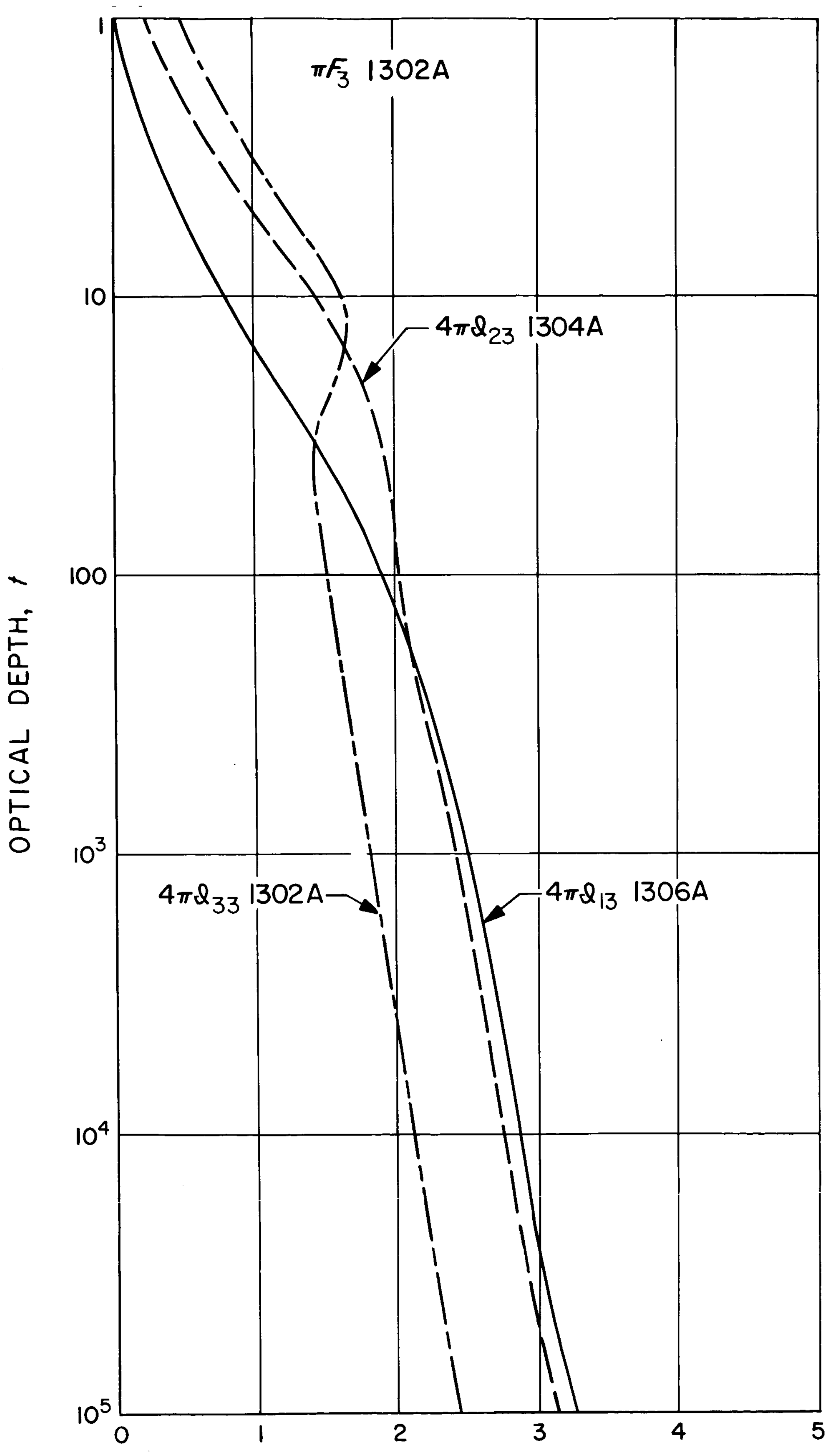
B_q
CASCADING
A_p
X_n
X_ℓ
X_m
F_{qn}
F_{pm}



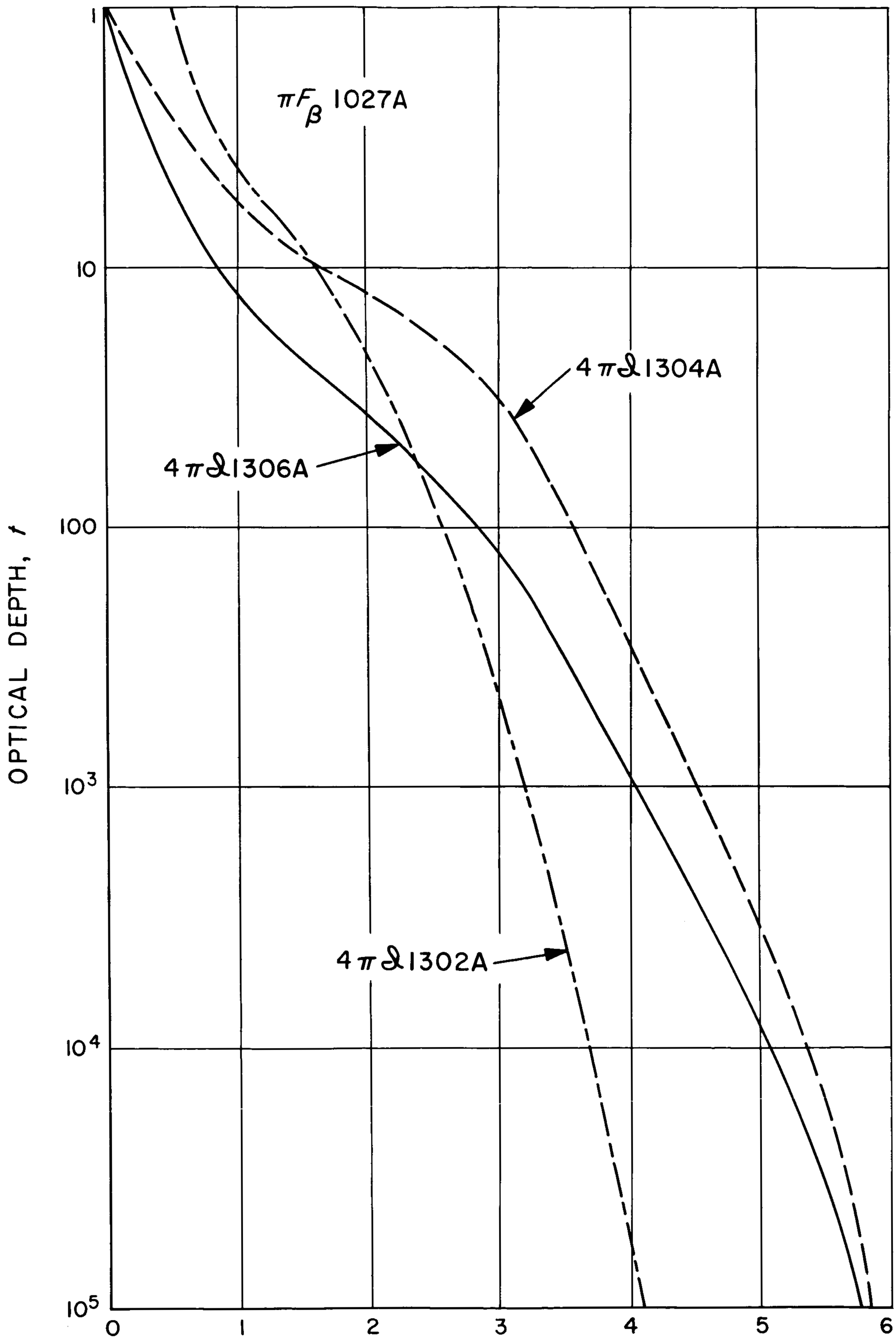
OPTICAL DEPTH, τ
1
10
100
1000
10^4
10^5
0
1
2
3
4
5
6
7
8
9
10
11
πF_1 1306A
$4\pi\mathcal{J}_{21}$ 1304A
$4\pi\mathcal{J}_{11}$ 1306A
$4\pi\mathcal{J}_{31}$ 1302A



πF2 1304A
4πϑ22 1304A
4πϑ12 1306A
4πϑ32 1302A
OPTICAL DEPTH, τ
1
10
100
10³
10⁴
10⁵
0
1
2
3
4
5
6



πF_3 1302A
$4\pi \mathcal{l}_{23}$ 1304A
$4\pi \mathcal{l}_{33}$ 1302A
$4\pi \mathcal{l}_{13}$ 1306A
OPTICAL DEPTH, t
1
10
100
10^3
10^4
10^5
0
1
2
3
4
5



πF_β 1027A
$4\pi\mathcal{J}$ 1304A
$4\pi\mathcal{J}$ 1306A
$4\pi\mathcal{J}$ 1302A
OPTICAL DEPTH, t
1
10
100
10^3
10^4
10^5
0
1
2
3
4
5
6